

Spike Timing Control of Oscillatory Neuron Models Using Impulsive and Quasi-Impulsive Charge-Balanced Inputs

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Abstract—We propose a method to control the spike timing of a Type II oscillatory neuron to match the phase of a given reference oscillator. The control method is inspired by the impulsive character of neural communication in nature, and leads to a simple mathematical solution. We show that the phase response curve, which describes the phase-shift of the oscillation due to an impulsive perturbation as a function of the phase at which the perturbation occurs, contains sufficient information to design a charge-balanced control law that provides global monotonic convergence of oscillator phase to the reference phase. This feedback law requires only the knowledge of the dynamics gained through the phase reduction, and the ability to detect a once-per-period marker event, such as the time at which a neuron fires. The effectiveness of this control law is demonstrated through analytical and numerical results, including application to the full-dimensional conductance-based neuron model from which the phase-reduced model was derived. This work represents a step toward a closed-loop form of electrical deep brain stimulation, a treatment for neuromotor disorders such as Parkinson’s disease, with symptoms characterized by pathologically synchronized neural firing.

I. INTRODUCTION

In this paper, we consider event-based control of a nonlinear phase oscillator, derived from an oscillatory neuron model, subject to the constraint that the integral of the control signal over a period of actuation is zero, and the only observable is a once-per-period marker event. In previous work [6], we developed several control strategies for a simplified phase oscillator and proposed a heuristic extension to phase oscillator models representing real neurons. Here, we propose a more rigorous control method, inspired by the signaling dynamics of real neurons, that is applicable to a rather large class of nonlinear phase oscillators. We show that it provides global monotonically convergent error dynamics when applied to full-dimensional conductance-based neuron models, in addition to the phase-reduced models.

This problem is motivated by the desire to control the spiking behavior of oscillatory neurons and, by extension, the level of synchronization of a population of oscillatory neurons. For example, one of the symptoms of Parkinson’s Disease (PD) is tremor in the limbs, which has been associated with pathological synchronization of motor control neurons in the thalamus [10]. These symptoms can be treated using a procedure known as Electrical Deep Brain

Stimulation (EDBS). Current technology uses an open loop high-frequency (greater than 100 Hz) waveform directed into the brain through an implanted electrode by a device similar to a cardiac pacemaker [2]. Incorporating a sensor electrode to provide feedback to the stimulus generator has been suggested as a way to optimize the efficacy of the treatment and reduce its side effects via “Demand-Controlled EDBS” [12]. Ultimately, we envision a set of feedback electrodes and a set of stimulus electrodes working with a controller incorporated into the EDBS signal generator electronics. The controller would use measurements of the neural population to derive optimal stimulus signals to accomplish population-level control objectives, such as desynchronization in the case of PD.

From a control theory standpoint, such a population of millions of neurons, each connected to tens-of-thousands of other neurons, is a staggeringly complex and perhaps intractable system. The neurons themselves are highly nonlinear, display a large degree of variability, and the network structure is unknown (and perhaps unknowable). However, since open-loop EDBS has been shown to be effective in eliminating tremors in PD, epilepsy, and even symptoms of depression [2], [9], there is good reason to believe that there may be some inherent structure, important to the dynamics of synchronization, that may enable the problem to be cast in a tractable form.

While the intent of this paper is to derive control algorithms for individual oscillatory neurons, these results can be interpreted in the larger context of population-level control. Indeed, if we drastically simplify the problem by neglecting coupling and assume that we can sense and stimulate each neuron (or perhaps each cluster of neurons) independently, we can obviously desynchronize this simple neural system by driving each neuron (or cluster) to follow a uniformly staggered reference phase. While this toy model is not very biologically realistic, we hope that the methods which we have developed for individual neurons will be a starting foundation for future work to address more realistic network architectures.

II. PROBLEM STATEMENT

Our control objective is to drive an oscillatory neuron to track the phase of a reference trajectory evolving at a constant frequency (equal to the natural frequency of the neuron), using a charge-balanced control signal based only on the detection of a voltage spike.

To model oscillatory neurons, we start with a set of ordinary differential equations (ODEs) obtained from rep-

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representing the neuron as an electrical circuit, as originally developed in the seminal work of Hodgkin and Huxley [7]. We consider regions of the ODE parameter space where the system exhibits a stable periodic orbit. Then, we perform a phase reduction to arrive at a one-dimensional nonlinear phase oscillator model, which has a single state on the unit circle with zero phase identified as the voltage spike. This reduction provides the necessary information to develop the desired control algorithms.

A. Neuron models

The membrane voltage dynamics of neurons are represented using the conductance-based Hodgkin-Huxley formalism [7], in space-clamped form, yielding a set of ODEs of the form:

$$c\dot{V} = \mathbf{I}_g(V, \mathbf{n}) + I_b + I(t), \quad \dot{\mathbf{n}} = \mathbf{G}(V, \mathbf{n}), \quad (1)$$

where $V \in \mathbb{R}$ is the voltage across the membrane, $\mathbf{n} \in \mathbb{R}_{[0,1]}^m$ is the vector of gating variables which correspond to the state of the membrane's ion channels, $c \in \mathbb{R}^+$ is the constant membrane capacitance, $\mathbf{I}_g : \mathbb{R} \times \mathbb{R}^m \mapsto \mathbb{R}$ is the sum of the membrane currents, and $I : \mathbb{R} \mapsto \mathbb{R}$ is the stimulus current. $I_b \in \mathbb{R}$ is the baseline current, which is a bifurcation parameter that controls whether the neuron is in an excitable or an oscillatory region. This form of ODE representation was first employed by Hodgkin and Huxley to model the *Loligo* squid's giant axon [7].

This canonical model, while not representing brain neurons, is the prototypical voltage membrane dynamic model, and exhibits similar characteristic oscillatory behavior as the motor control neurons that we are interested in. Due to this qualitative similarity, and the fact that the Hodgkin-Huxley model is perhaps the most widely studied and familiar conductance-based neuron model, we choose to consider it as our primary neuron model in this paper.

Control of the spike timing of this system is non-trivial due to the fact that the vector functions \mathbf{I}_g and \mathbf{G} are highly nonlinear, the gating variables \mathbf{n} are not observable by the controller, and equations themselves represent a feedback interconnection of nonlinear systems. For the Hodgkin-Huxley model we consider, the gating variables \mathbf{n} represent Sodium and Potassium ion channels, and a generic leakage channel. The functions composing \mathbf{G} consist of sums of ratios of exponentials derived by curve-fitting experimental data. The details of this model can be found, for example, in the appendix of [4]. We seek a simpler representation of the dynamics of this model that will capture its fundamental behavior, but be amenable to analysis.

B. Phase reduction

In the case of oscillatory synchronization, the participating neurons fire periodically, corresponding to a region of I_b parameter space where the ODEs (1) have a stable periodic orbit. We will consider the case when $I_b = 10$ mA. In the absence of stimuli, the system is found to have a stable periodic orbit with natural frequency $\omega = 0.43$ rad/msec [11]. We then perform a phase reduction, following [4], to

reduce the number of dimensions from $m + 1$ ($m = 3$, in the case of the Hodgkin-Huxley model) to a single phase variable.

We define $\mathbf{x} \equiv [V, \mathbf{n}^T]^T$ so that we can conveniently represent the entire state of the full-dimensional model in one vector. We introduce the phase variable $\theta \in \mathbb{S}^1$ which parametrizes the position of the state on its periodic orbit $\mathbf{x}^\gamma(\theta)$. In the absence of input, the system simply evolves with constant frequency ω along $\mathbf{x}^\gamma(\theta)$. With nonzero input, the system continues to evolve due to ω , but is affected by the input depending on where on the periodic orbit the state is. In general, the phase-reduced dynamics obey the following ODE

$$\dot{\theta} = \omega + \mathbf{Z}(\theta) \cdot \mathbf{u}(t),$$

where the phase response curve $\mathbf{Z}(\theta)$ and the input $\mathbf{u}(t)$ are vector functions of the same dimension as the original system (1). However, since the electrical stimulus $I(t)$ affects only the voltage direction, $\mathbf{u}(t) = [I(t)/c, 0, 0, 0]^T$, and we can ignore the three components of $\mathbf{Z}(\theta)$ corresponding to the gating variables \mathbf{n} . Thus we obtain the phase-reduced model

$$\dot{\theta} = \omega + Z(\theta)u(t), \quad (2)$$

where $Z(\theta) = \mathbf{Z}_V(\theta)$ is the voltage component of the phase response curve, and $u(t)$ is the input current $I(t)$ normalized by the capacitance c . The phase-reduced model is valid in a neighborhood of the periodic orbit where perturbations off \mathbf{x}^γ are asymptotically attracted back with a phase-shift dictated by isochrons, as described in [13] and summarized in [4].

In this paper, we will consider only Type II neurons [8], i.e. those with phase response curves derived from systems exhibiting a Hopf (or Bautin) bifurcation [4]. The Hodgkin-Huxley model yields such a phase response curve, which is shown in Figure 1 and labeled with the following important points:

$$\begin{aligned} \alpha &= \operatorname{argmin}(Z(\theta)) & , & \quad Z_{min} = Z(\alpha) \\ \beta &= \operatorname{argmax}(Z(\theta)) & , & \quad Z_{max} = Z(\beta). \end{aligned}$$

In general, phase response curves derived from Type II neurons yield a class of nonlinear phase oscillators with phase response curves characterized by the following conditions:

$$\begin{aligned} Z(0) &= 0 & , & \quad Z'(0) < 0 \\ Z(\gamma) &= 0 & , & \quad Z'(\gamma) > 0 \\ Z_{max} &> 0 & , & \quad Z_{min} < 0 \\ &0 < \alpha < \gamma < \beta < 2\pi & . \end{aligned} \quad (3)$$

C. Control objective

The events of critical importance in neural signaling and synchronization are the voltage spikes [10] which we define to occur at $\theta = 0$ for our phase-reduced model. Our control objective is drive the neuron to spike (pass through $\theta = 0$) in-phase with a reference trajectory with the same natural frequency.

A complicating factor is the fact that the only observable is the membrane voltage, since the gating variables are non-physical abstractions of the microscopic states of the ion

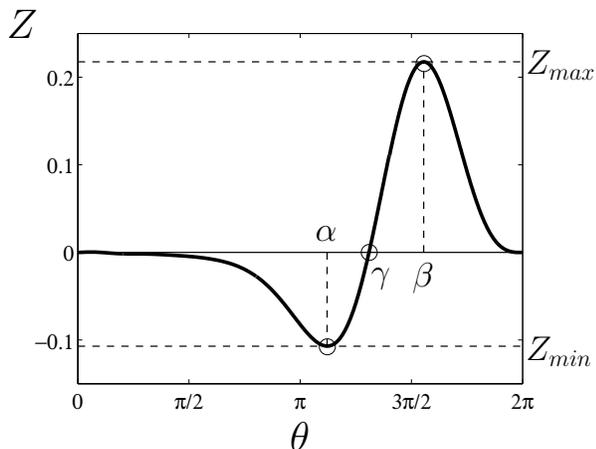


Fig. 1. Phase response curve derived from the Hodgkin-Huxley neuron model for $I_b = 10\text{mA}$ [11].

channels in the cell membrane. The relationship between voltage and phase is not one-to-one, so phase cannot be ascertained only by voltage except when the neuron spikes. So rather than developing a control law based on continuous voltage feedback, we will focus on observing the spikes as *events*. Much work on event-based control of nonlinear systems has been done, especially with respect to stochastic processes [1]. While we do not consider stochasticity here, the event-based framework is well-suited to our control problem, since $Z(\theta)$ is non-invertible and traditional feedback linearization fails.

When a spike event happens, we compare the neuron's phase to that of its reference oscillator, and construct an open-loop waveform that will actuate the neuron with the goal of correcting all, or a portion of, its phase error by the time the neuron spikes again. The reference oscillator evolves according to the simple equation

$$\theta_r(t) = \omega_r t + \theta_r(0) \pmod{2\pi}$$

where $\omega_r = \omega$ is the natural frequency of the reference oscillator, and $\theta_r(0)$ is its initial condition. The times at which the reference oscillator crosses zero are the times we want the controlled phase oscillator (neuron) to cross zero (spike).

Generally, one would define the phase error as

$$\Delta\theta = \theta - \theta_r.$$

In the scenario presented in this paper, the phase error is sampled only when $\theta = 0$, so effectively $\Delta\theta = -\theta_r$. However, the phase error, as presently defined, exists on $(-2\pi, 0]$. To accommodate convenient notation for later development, we wrap the phase error to the interval $(-\pi, \pi]$. This is accomplished using the following phase wrapping algorithm (shown here in general form):

$$\Delta\theta = \begin{cases} \theta - \theta_r, & \text{for } |\theta - \theta_r| \leq \pi \\ \theta - \theta_r - \text{sgn}(\theta - \theta_r)2\pi, & \text{for } |\theta - \theta_r| > \pi \end{cases} \quad (4)$$

We are interested in how $\Delta\theta$ changes after a period of control actuation, so we will define $\Delta\theta^+$ to be the phase error at the time of the next spiking event. Thus, we seek a control law that decreases the phase error over one period, i.e.

$$\Delta\theta^+ / \Delta\theta < 1 \quad \forall \Delta\theta \in (-\pi, \pi]$$

excluding $\Delta\theta = 0$ (where $\Delta\theta^+$ should also equal 0).

Additionally, we desire a control algorithm that provides actuation signals that are small in magnitude and does not result in charge accumulation (the integral of the control signal over one period of actuation should equal zero), in order to minimize collateral damage to the neuron and its surrounding tissue in the brain.

III. IMPULSIVE CONTROL

In nature, neurons communicate by voltage spikes that are large in magnitude but very short in duration. Signals of this kind are a biological analog of impulses. In fact, dynamical systems researchers in mathematical neuroscience have long used the concept of impulsive coupling to model networks of neurons [3]. This has inspired the idea of using impulsive signals (Dirac delta functions) for spike timing control. Impulses are desirable inputs from the perspective of the phase-reduced model, since delta functions turn the calculus into simple algebra. For example, consider the dynamics of generic phase-reduced model over the time interval $[t_I, t_{II}]$ subject to an impulsive input at time t^* :

$$\dot{\theta} = \omega + Z(\theta)\tilde{u}\delta(t - t^*)$$

where the $t_I \leq t^* < t_{II}$. The solution is simply

$$\theta(t_{II}) = \theta(t_I) + \omega(t_{II} - t_I) + Z(\theta(t_I) + \omega(t^* - t_I))\tilde{u} \pmod{2\pi}. \quad (5)$$

We will proceed by using (5) as the basic building block of our control scheme. Intuitively, we want to drive the oscillator with impulses timed to occur when its phase corresponds to that of the extremal values of its phase response curve. For example, if the control objective is to speed up the neuron, the optimum strategy is stimulate with a negative impulse, timed to occur when $\theta = \alpha$ (recall $Z(\alpha) = Z_{min} < 0$), followed by a positive impulse, timed to occur when $\theta = \beta$ (recall $Z(\beta) = Z_{max} > 0$).

Since we are not considering noise, we can use (5) to predict the phase of the actuated oscillator using simple algebra. The charge balance constraint is implemented by simply constraining the control to be in the form of two timed impulses of equal magnitude but opposite sign. Recall that the control objective is to reduce the phase error after each period of actuation, i.e. $\Delta\theta^+ < \Delta\theta$. The following control algorithm, derived using (5), gives $\Delta\theta^+ = K\Delta\theta$, where we choose $K \in [0, 1)$:

$$u(t) = \tilde{u}(\delta(t - t_1) - \delta(t - t_2)), \quad (6)$$

where

$$\tilde{u} = \frac{(1 - K)\Delta\theta}{Z_{max} - Z_{min}}, \quad t_1 = \frac{\alpha}{\omega}, \quad t_2 = \frac{1}{\omega}(\beta - Z_{min}\tilde{u}). \quad (7)$$

Condition	Minimum admissible correction factor K_{min}
$0 < \theta(t_1^+)$	$1 + \frac{\alpha(Z_{max} - Z_{min})}{\pi Z_{min}}$
$\theta(t_1^+) < \gamma$	$1 + \frac{(\gamma - \alpha)(Z_{max} - Z_{min})}{\pi Z_{min}}$
$\gamma < \theta(t_2^+)$	$1 - \frac{(\beta - \gamma)(Z_{max} - Z_{min})}{\pi Z_{max}}$
$\theta(t_2^+) < 2\pi$	$1 - \frac{(2\pi - \beta)(Z_{max} - Z_{min})}{\pi Z_{max}}$

TABLE I
CONSTRAINTS ON K_{min}

In the control schemes presented in this paper, t represents the time since the last spiking event and is reset to zero whenever θ crosses the $\theta = 2\pi = 0$ spike threshold. While this is somewhat of an abuse of notation, it will greatly simplify the presentation of the control schemes by relinquishing the need to carry an incrementing time offset.

In terms of the cost function $J = \int_0^\infty |u(t)| dt$ this impulsive control law is obviously optimal, since it corrects the error exactly as we intend over one control period, and any other control waveform would be using energy at a “weaker” part of the phase response curve, or would violate the charge balance constraint.

We must remember, however, that the phase-reduced model is a simplified representation of a higher-dimensional conductance-based model, and has a weakness that must be addressed. The behavior of the phase-reduced model is not necessarily representative of the conductance-based model when the impulses are large enough to drive the oscillator to a phase where the sign of the phase response curve is different from what it was prior to the impulse. Also, if the oscillator is driven beyond the $\theta = 2\pi = 0$ spike threshold, in either direction, the phase-reduced model loses relation to the conductance-based model, since a phase of zero implies a firing and an essential “reset” of the oscillator.

Since we are concerned with asymptotic convergence to a fixed frequency reference trajectory, we can easily avoid these issues by using fractional error correction with

$$K \geq K_{min} > 0,$$

where K_{min} is determined by the phase response curve $Z(\theta)$. Table I lists the corner conditions for K_{min} . In the table, $\theta(t_1^+)$ refers to the phase of the oscillator immediately after the impulse at $t = t_1$. Likewise $\theta(t_2^+)$ refers to the phase immediately after the impulse at $t = t_2$.

IV. QUASI-IMPULSIVE CONTROL

We now modify the impulsive control to use finite magnitude and nonzero duration control pulses. Using finite (small) magnitude control pulses is important in the context of EDBS, since the brain tissue exposed to the electrical stimulus can be damaged by large electrical currents. Also, the phase reduction method assumes that the input acts as a small perturbation. A digital approximation of a Dirac

delta function as a rectangular spike with magnitude \tilde{u}/dt , where dt is equal to the sample time, works well for numerical simulation of the phase-reduced nonlinear oscillator model, but is inappropriate for use with the full-dimensional conductance-based model. It can instantaneously jolt the state far off its periodic orbit and yield results that are not closely approximated by the phase reduced model.

To address these issues, we develop a quasi-impulsive control that uses the same control energy as the impulsive control, but extends the duration and confines the magnitude of the impulse to be equal to a threshold C , which is chosen to be greater than or equal to a certain minimum value C_{min} . Obviously, we are sacrificing optimality, since the finite duration pulses may be stimulating the neuron at sub-extremal regions of the phase-response curve. We will show, however, that when implemented on the full-dimensional Hodgkin-Huxley neuron model, the resulting performance is quite close to the fractional error correction factor K derived from the optimum impulsive control method (6).

Analogous to a delayed bang-bang control, this method stimulates at magnitudes equal to the threshold constraint $C \geq C_{min}$, using rectangular pulses of opposite sign centered at t_1 and t_2 with durations such that the integral of each pulse is equal to \tilde{u} . Using a value of fractional error correction $K \geq K_{min}$ satisfying the conditions listed in Table I, we propose the following control scheme:

$$u(t) = \begin{cases} 0 & , \text{ for } 0 \leq t < t_A \\ \text{sgn}(\Delta\theta)C & , \text{ for } t_A \leq t < t_B \\ 0 & , \text{ for } t_B \leq t < t_C \\ -\text{sgn}(\Delta\theta)C & , \text{ for } t_C \leq t < t_D \\ 0 & , \text{ for } t_D \leq t \end{cases} \quad (8)$$

where

$$\begin{aligned} t_A &= t_1 - \frac{|\tilde{u}|}{2C} & , & \quad t_B = t_1 + \frac{|\tilde{u}|}{2C} \\ t_C &= t_2 - \frac{|\tilde{u}|}{2C} & , & \quad t_D = t_2 + \frac{|\tilde{u}|}{2C} \end{aligned} \quad (9)$$

and \tilde{u} is as defined previously in (7). The corner conditions that determine the minimum admissible threshold constraint C_{min} are listed on Table II. These conditions are derived in similar manner as those for the minimum fractional error correction factor K_{min} . Together, these constraints ensure that the control signal always stimulates in the right direction and will yield a charge balanced waveform. In the limit of $C \rightarrow \infty$, this scheme recovers the timing of the purely impulsive control law (6).

Theorem *For the phase-reduced neural oscillator model $\dot{\theta} = \omega + Z(\theta)u(t)$ where $u(t)$ is as defined in (8), $Z(\theta)$ satisfies the conditions from (3), K satisfies the conditions in Table I, and C satisfies the conditions in Table II, the phase error ratio over one period of actuation will be a strict contraction ($\frac{\Delta\theta^+}{\Delta\theta} < 1$), implying global monotonic convergence of the oscillator phase $\theta(t)$ to the reference phase $\theta_r(t)$.*

Condition	Minimum admissible control magnitude C_{min}
$0 < t_A$	$\frac{\omega\pi(1-K)}{2\alpha(Z_{max}-Z_{min})}$
$t_B < t_C$	$\max_{\Delta\theta \in (-\pi, \pi]} \left(\frac{\omega\Delta\theta(1-K)}{(\beta-\alpha)(Z_{max}-Z_{min})-Z_{min}(1-K)\Delta\theta} \right)$
$\theta(t_B) < \gamma$	$\max_{\Delta\theta \in (-\pi, \pi]} \left(\frac{-\omega(1-K)\Delta\theta}{2[(\gamma-\alpha)(Z_{max}-Z_{min})-Z_{min}(1-K)\Delta\theta]} \right)$
$\theta(t_C) > \gamma$	$\frac{\omega\pi(1-K)}{2(\beta-\gamma)(Z_{max}-Z_{min})}$
$\theta(t_D) < 2\pi$	$\max_{\Delta\theta \in (-\pi, \pi]} \left(\frac{-\omega(1-K)\Delta\theta}{2[(2\pi-\beta)(Z_{max}-Z_{min})+Z_{max}(1-K)\Delta\theta]} \right)$

TABLE II
CONSTRAINTS ON C_{min}

Proof: First, a word on notation. When developing bounds to prove error convergence, underbars \underline{x} and overbars \bar{x} will denote the greatest lower and least upper bounds on the variable x , respectively. The objective of the proof is to show that the error gain, $\frac{\Delta\theta^+}{\Delta\theta}$, is strictly less than one for all values of initial error $\Delta\theta \in (-\pi, \pi]$. This implies that the phase error is reduced after each event-driven actuation period. And since the oscillator in absence of input rotates around \mathbb{S}^1 with natural frequency ω , events are persistent in time, which make it impossible for a steady state error to exist.

If the extension of the impulsive control to the quasi-impulsive case were perfect, we would expect $\frac{\Delta\theta^+}{\Delta\theta} = K$. This, however, is the greatest lower bound, since a pulse with nonzero duration implies that the control will be stimulating the neuron at phases where the phase response curve may be sub-extremal. We will proceed with the proof by developing bounds on the time at which the oscillator will spike (cross the $\theta = 2\pi = 0$ threshold), which we will denote as t^+ , and which will be compared with the time at which the constant frequency reference oscillator spikes to determine the phase error after one period of actuation, $\Delta\theta^+$.

For simplicity, we will develop bounds on t^+ by separately considering the cases $\Delta\theta > 0$ and $\Delta\theta < 0$. When $\Delta\theta = 0$, no control action is taken so that $\Delta\theta^+ = 0$.

Case I: $\Delta\theta > 0$

Intuitively, the control should slow the neuron down when $\Delta\theta > 0$. A control magnitude $C \geq C_{min}$ satisfying the conditions in Table II guarantees that throughout the duration of the first pulse, the oscillator will have a phase between 0 and γ , the region where $Z(\theta)$ is negative semidefinite. For a positive $\Delta\theta$, the pulse will be positive, so the stimulus can only decrease the velocity of the oscillator below ω . Likewise, admissibility of the control magnitude further guarantees that the oscillator's phase will be between γ and 2π (the region where $Z(\theta)$ is positive semidefinite) during the second pulse which is negative, since $\Delta\theta > 0$. Again this means that the control signal can only decrease the oscillator's velocity below its natural frequency ω .

If there was no control, the neuron would spike again at $t^+ = 2\pi/\omega$, which would result in $\Delta\theta^+ = \Delta\theta$. In view of

the argument above, this is, in fact, the lower bound \underline{t}^+ .

Now we step through the dynamics to develop an upper bound for t^+ . We begin at $\theta(0) = 0$. Then, advancing with zero input until t_A ,

$$\theta(t_A) = \omega t_A = \alpha - \frac{\omega(1-K)\Delta\theta}{2C(Z_{max}-Z_{min})}.$$

Now we calculate a lower bound on $\theta(t_B)$. We do this by using Z_{min} as a lower bound on the phase response curve. Between t_A and t_B , our input is equal to C . We obtain

$$\begin{aligned} \underline{\theta}(t_B) &= \theta(t_A) + (\omega + Z_{min}C)(t_B - t_A) \\ &= \alpha + \frac{(\omega + 2CZ_{min})(1-K)\Delta\theta}{2C(Z_{max}-Z_{min})}. \end{aligned}$$

We then evolve with zero input until t_C :

$$\underline{\theta}(t_C) = \underline{\theta}(t_B) + \omega(t_C - t_B) = \beta - \frac{\omega(1-K)\Delta\theta}{2C(Z_{max}-Z_{min})}.$$

The input is then applied again, this time with in the negative direction, since we wish to slow the neuron down, and $Z_{max} > 0$. We obtain

$$\begin{aligned} \underline{\theta}(t_D) &= \underline{\theta}(t_C) + (\omega - Z_{max}C)(t_D - t_C) \\ &= \beta + \frac{(\omega - 2CZ_{max})(1-K)\Delta\theta}{2C(Z_{max}-Z_{min})}. \end{aligned}$$

We now solve for the upper bound \bar{t}^+ using the relation

$$\theta(\bar{t}^+) = 2\pi = \underline{\theta}(t_D) + \omega(\bar{t}^+ - t_D),$$

giving

$$\bar{t}^+ = t_D + \frac{2\pi - \underline{\theta}(t_D)}{\omega} = \frac{2\pi + (1-K)\Delta\theta}{\omega}$$

So for $\Delta\theta > 0$,

$$\frac{2\pi}{\omega} < t^+ \leq \frac{2\pi + (1-K)\Delta\theta}{\omega}.$$

In terms of phase, these bounds on t^+ imply $K \leq \frac{\Delta\theta^+}{\Delta\theta} < 1$, as desired.

Case II: $\Delta\theta < 0$

When $\Delta\theta < 0$, the control method seeks to speed up the oscillator. Following the $C \geq C_{min}$ admissibility argument from Case I, but with the signs flipped, we conclude that the control signal cannot slow the oscillator down. Thus we have a simple upper bound: $\bar{t}^+ = 2\pi/\omega$. We can now step through the dynamics in the same manner as Case I, but with $\Delta\theta < 0$, to yield the inequality

$$\frac{2\pi + (1-K)\Delta\theta}{\omega} \leq t^+ < \frac{2\pi}{\omega}.$$

Therefore, $K \leq \frac{\Delta\theta^+}{\Delta\theta} < 1$, as claimed.

Thus, for all nonzero values of $\Delta\theta \in (-\pi, \pi]$, the control provides error contraction over one period of actuation, and if $\Delta\theta = 0$, the control takes no action. ■

The solid lines in the plots in Figure 2 illustrate the performance of this control algorithm for the phase-reduced model derived from the Hodgkin-Huxley system with the phase response curve shown previously in Figure 1. For this phase response curve, the minimum admissible values K_{min} and C_{min} are 0.63 and 1.65 mA, respectively. The results shown are for $K = 0.7$ and $C = 1.7$ mA. We see that the gain $\frac{\Delta\theta^+}{\Delta\theta}$ is between 0.7 and 0.8 over the entire interval, quite close to our prescribed K value of 0.7 derived from the optimal impulsive control method (6). As discussed in [6], the global stability of the origin of $M : \Delta\theta \mapsto \Delta\theta^+$ determines the global asymptotic stability of the phase error. Here, M is well-behaved, smooth, and is confined to the first and third quadrants (as expected with global monotonic convergence).

V. APPLICATION TO THE FULL-DIMENSIONAL MODEL

We now implement the quasi-impulsive control method (8) on the full-dimensional neuron model (1) using parameters listed in [4]. Our objective is to show that the phase error gain $\frac{\Delta\theta^+}{\Delta\theta}$ is less than one for all initial values of $\Delta\theta$. We will also compare the results to those achieved with the phase-reduced model.

Before outlining our results, we will briefly explain how we implement the control, which was developed for the phase-reduced model, on the full-dimensional model. For a single simulation, we choose an initial error $\Delta\theta$. We initialize the state of the model with phase $\theta(0) = 0$ (the state vector representation of that point on the periodic orbit, $\mathbf{x}(0)$, is known based on information derived during the phase reduction). We then integrate the ODE system (in \mathbf{x} coordinate space) using the electrical stimulus signal $I(t) = cu(t)$, where we recall that c is the constant membrane capacitance (which for the standard Hodgkin-Huxley system is equal to $1.0\mu\text{F}/\text{cm}^2$). The simulation proceeds until a spike is detected (the details of spike detection and phase sampling can be found in [5]). The timing of this spike is compared to the timing of the reference oscillator spike (initialized based on the choice of $\Delta\theta$) to obtain the value of $\Delta\theta^+$. The results of fifty individual simulations with initial conditions ranging over $\Delta\theta \in (-\pi, \pi]$ are shown as a black line with white circle markers on Figure 2. We see that the implementation of the control law based on the phase-reduced model yields very similar results for the full-dimensional system. These results represent a significant improvement over previous work [6]. Here we have monotonic error convergence, whereas previous methods yielded asymptotic error convergence of $|\frac{\Delta\theta^+}{\Delta\theta}|$, a somewhat weaker control objective.

VI. CONCLUSIONS

We have developed a method to control the spike timing of a phase-reduced model of an oscillatory neuron that provides global monotonic convergence of the oscillator's phase to that of a reference phase trajectory with the same natural frequency. We have shown that the resulting control law is effective in achieving the same objective when applied to

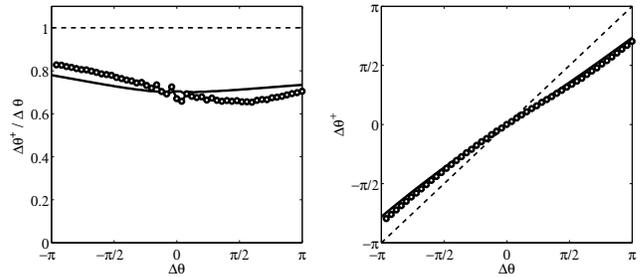


Fig. 2. Quasi-impulsive control algorithm performance. The left plot shows the phase error gain. The right plot shows the $\Delta\theta \mapsto \Delta\theta^+$ map. Solid lines are results from the phase-reduced model to be compared with the white circle markers, which are results from the full-dimensional Hodgkin-Huxley system.

the full-dimensional conductance-based neuron model. The control algorithms developed in this paper are applicable to any Type II oscillatory neuron, and have been illustrated on the prototypical Hodgkin-Huxley model.

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